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SECOND-ORDER EFFECTS IN THE PROPAGATION OF ELASTIC WAVES*

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Abstract—Second-order effects in the propagation of elastic waves through homogeneous isotropic media are considered. The jump conditions governing second-order quantities at wave fronts where the first-order displacements have discontinuous derivatives are derived, and it is seen that a consistent theory requires the second-order displacements to be discontinuous at such wave fronts, in general. The results are illustrated by some one-dimensional examples involving plane and spherical waves.

1. INTRODUCTION

THE PAPER considers second-order effects arising in the propagation of elastic waves through a homogeneous isotropic medium under the assumption that for some interval of time all field quantities admit expansions in a small parameter ε representing the magnitude of the deformation. The governing equations for the second-order displacements are given in Section 3 and they are of the same form as those of the classical theory of elasticity but they contain inhomogeneous terms which arise from the first-order or infinitesimal displacements. A simple one-dimensional example is given in Section 5 in order to illustrate the occurrence of resonance in the second-order effects. The example also serves to illustrate the limiting form of the second-order solution for a sequence of first-order displacements which approach a field with a sharp wave front. The main portion of the paper, Section 6, discusses the second-order displacements associated with first-order displacements produced by waves with sharp fronts across which derivatives of the first-order displacements are discontinuous. It is shown that a theory consistent with the second-order field equations requires the second-order *displacements* to be discontinuous in general across such sharp wave fronts. The relations connecting the discontinuities in the second-order displacements with the discontinuities in the first derivatives of the classical displacements are determined from the governing differential equations by means of an integration and a limiting process. A direct derivation of these second-order jump relations from the exact (non-linear) conditions of mass and momentum balance across a wave front (see [1], for example) does not appear to be possible in a simple way, because the location and speed of the actual wave front are dependent on ε but the second-order quantities (i.e. the coefficients of ε^2 in the expansions) are assumed to be independent of ε. Thus the perturbation method used here provides the same constant wave speeds for the second-order displacements as for the first-order displacements.

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An alternative method has been given by Davison [2] for one-dimensional wave propagation in which a correction is made to the wave speed for the second-order terms. Exact treatments of finite amplitude, one-dimensional, elastic waves may be found in [2] and in [3], where additional references are available.

2. BASIC EQUATIONS

We consider a homogeneous elastic body which is isotropic in its undeformed state. At time $t = 0$, the initial time, the body is undeformed and the coordinates of a typical point referred to a fixed rectangular Cartesian coordinate system x are denoted by X_i, where Latin indices take the values 1, 2, and 3. Subsequently the body is deformed and the coordinates of a typical point in the system x at time t are denoted by x_i. We can use either the particle coordinates X_i or the spatial coordinates x_i together with the time t as independent variables, although the analysis presented in this paper employs, for the most part, x_i and t as free variables.

The components of the displacement vector at time t will be denoted by either $U_i(X_j, t)$ or $u_i(x_j, t)$. Our notation for gradients and time derivatives is:

$$U_{i,j} = \frac{\partial U_i(X_m, t)}{\partial X_j}, \qquad \dot{U}_i = \frac{\partial U_i(X_j, t)}{\partial t},$$

$$u_{i,j} = \frac{\partial u_i(x_m, t)}{\partial x_j}, \qquad \dot{u}_i = \frac{\partial u_i(x_j, t)}{\partial t}.$$

The stress components referred to the coordinate system x and measured per unit area of the deformed body are written as $\Sigma_{ij}(X_m, t)$ or $\sigma_{ij}(x_m, t)$.

The equations of motion in the absence of body forces are

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \ddot{U}_i, \tag{2.1}$$

where ρ is the density of the deformed body, and a repeated index implies summation over the values 1, 2, and 3.

Conservation of mass yields

$$\rho_0 = \rho\tau, \tag{2.2}$$

where ρ_0 is the density of the undeformed body and

$$\tau = \det(\delta_{ij} + U_{i,j}) = \{\det(\delta_{ij} - u_{i,j})\}^{-1}. \tag{2.3}$$

If the strain energy per unit volume of the undeformed body is denoted by $\Sigma(U_{i,j})$, then stress and strain energy are connected in a compressible material by the equation

$$\Sigma_{ij}(X_m, t) = \frac{1}{\tau}(\delta_{jm} + U_{j,m})\frac{\partial \Sigma}{\partial (U_{i,m})}, \tag{2.4}$$

and for an incompressible material by the equation

$$\Sigma_{ij}(X_m, t) = (\delta_{jm} + U_{j,m})\frac{\partial \Sigma}{\partial (U_{i,m})} - \Pi\delta_{ij}, \tag{2.5}$$

where $\Pi(X_i, t) = \pi(x_i, t)$ represents the hydrostatic pressure term.

For our purposes, we require the most general form of the strain-energy function for an isotropic material if terms of order higher than the third in the displacement gradients are neglected, linear terms being omitted because the stresses are assumed to be zero in the undeformed state. Consequently, in a compressible material the second-order incomplete form for Σ is used (see [4], for example), while the corresponding expression in an incompressible medium is given by the Mooney form [5].

The field equations for the displacements can now be found by substituting the appropriate form for Σ into (2.4) or (2.5) and then putting the resulting expressions for the stresses into the equations of motion (2.1). For an incompressible material $\rho = \rho_0$, so that in addition we have the equation $\tau = 1$. In deriving these field equations, we can ignore consistently all terms of the third order and higher in the displacements and their derivatives, because we are concerned only with second-order effects here.

3. SECOND-ORDER EFFECTS

A basic assumption of the method of successive approximations (see [6], for example) is that the displacements can be expanded as absolutely convergent power series in a real parameter ε, the series being twice differentiable term by term. The validity of such a procedure for elastic equilibrium problems has been justified under certain assumptions (see [7, 8]), but a corresponding justification is lacking for dynamical problems. It is clear that, in general, the series expansions will converge in the dynamical case for only a limited range of time, say $0 \leq t \leq t_1$.

Since we wish to obtain second-order effects in elastodynamics, we will assume that the displacements can be represented by expansions in terms of ε of the form

$$\begin{aligned} U_i(X_j, t) &= \varepsilon V_i(X_j, t) + \varepsilon^2 V_i'(X_j, t) + o(\varepsilon^2), \\ u_i(x_j, t) &= \varepsilon v_i(x_j, t) + \varepsilon^2 v_i'(x_j, t) + o(\varepsilon^2), \end{aligned} \tag{3.1}$$

these expansions being valid for sufficiently small ε, for relevant ranges of the coordinates, and for an interval of time $0 \leq t \leq t_1$. In a particular case, it would seem that the smaller the parameter ε, the larger the time interval for which the second order theory will give useful results. Implicit in (3.1), but made explicit here, is the fact that the first-order displacements, V_i or v_i, and the second-order displacements, V_i' or v_i', are independent of ε. We further assume that the first and second derivatives of U_i and u_i can be obtained directly by differentiating (3.1), the error still remaining $o(\varepsilon^2)$.

Under these assumptions, the stress–displacement gradient relations for a compressible material show that the stress components also have expansions in powers of ε and we write

$$\begin{aligned} \Sigma_{ij}(X_m, t) &= \varepsilon T_{ij}(X_m, t) + \varepsilon^2 T_{ij}'(X_m, t) + o(\varepsilon^2), \\ \sigma_{ij}(x_m, t) &= \varepsilon t_{ij}(x_m, t) + \varepsilon^2 t_{ij}'(x_m, t) + o(\varepsilon^2). \end{aligned} \tag{3.2}$$

For an incompressible material, we assume in addition that the hydrostatic pressure has the expansions

$$\begin{aligned} \Pi(X_i, t) &= \varepsilon P(X_i, t) + \varepsilon^2 P'(X_i, t) + o(\varepsilon^2), \\ \pi(x_i, t) &= \varepsilon p(x_i, t) + \varepsilon^2 p'(x_i, t) + o(\varepsilon^2), \end{aligned} \tag{3.3}$$

these expansions being differentiable once term by term. As with the displacements, the first-order stress components T_{ij} or t_{ij}, the second-order stress components T_{ij}' or t_{ij}', the first-order hydrostatic pressure P or p, and the second-order hydrostatic pressure P' or p' are all independent of ε.

The equations governing first- and second-order quantities are found by substituting into the basic field equations the appropriate expansions in ε, collecting terms, and equating to zero, separately, the coefficients of ε and ε^2. Clearly this approach is possible only because, as previously stated, the first- and second-order quantities involved are independent of ε.

For a compressible material, the classical (or first-order) equations thus obtained are

$$\ddot{v}_i = (c^2 - s^2)v_{m,mi} + s^2 v_{i,mm}, \tag{3.4}$$

$$t_{ij} = \rho_0(c^2 - 2s^2)v_{m,m}\delta_{ij} + \rho_0 s^2(v_{i,j} + v_{j,i}), \tag{3.5}$$

while the second-order field equations are

$$\begin{aligned}
\rho_0\ddot{v}'_i = {} & \rho_0(c^2 - s^2)v'_{m,mi} + \rho_0 s^2 v'_{i,mm} - 2\rho_0\dot{v}_m\dot{v}_{i,m} \\
& + 2(a_5 - a_1)(v_{i,n}v_{n,mm} + v_{n,i}v_{n,mm}) \\
& + 2(a_1 + 4a_2 - 2a_3 - a_5)(v_{m,n}v_{m,ni} + v_{n,i}v_{m,mn}) \\
& + 4(a_1 + 2a_2 - a_3 - a_5)v_{m,m}v_{i,nn} \\
& + 2(2a_1 + 8a_2 - 2a_3 - a_5)v_{m,n}v_{n,mi} \\
& + 4(a_5 - 2a_1)v_{m,n}v_{i,mn} \\
& - 2(a_5 + 2a_3)v_{i,n}v_{m,mn} \\
& + 4(5a_3 + 12a_4 + a_5)v_{n,n}v_{m,mi}
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
t'_{ij} = {} & \rho_0(c^2 - 2s^2)v'_{m,m}\delta_{ij} + \rho_0 s^2(v'_{i,j} + v'_{j,i}) \\
& + 2(a_5 - a_1)v_{m,i}v_{m,j} \\
& + 2(a_5 - 2a_1)(v_{i,n}v_{n,j} + v_{j,n}v_{n,i} + v_{i,n}v_{j,n}) \\
& + 2(a_1 + 2a_2 - a_3 - a_5)v_{m,n}v_{m,n}\delta_{ij} \\
& + 2(3a_1 + 4a_2 - 2a_3 - 2a_5)(v_{i,j} + v_{j,i})v_{m,m} \\
& - 4(a_1 + 2a_2 - 3a_3 - 6a_4 - a_5)v_{m,m}v_{n,n}\delta_{ij} \\
& + 2(2a_1 + 4a_2 - a_3 - a_5)v_{m,n}v_{n,m}\delta_{ij},
\end{aligned} \tag{3.7}$$

where our notation for derivatives of v_i (or V_i) and v'_i (or V'_i) follows that of the previous section for the derivatives of u_i (or U_i). In equations (3.4)–(3.7), c and s are the classical wave speeds, a_1, a_2, a_3, a_4, a_5 are material constants, and

$$\rho_0 c^2 = 8a_2 = \lambda + 2\mu, \qquad \rho_0 s^2 = -2a_1 = \mu,$$

where λ and μ are Lamé's constants.

For an incompressible medium, the classical field equations are

$$\rho_0\ddot{v}_i = \rho_0 s^2 v_{i,mm} - \frac{\partial p}{\partial x_i}, \tag{3.8}$$

$$t_{ij} = -p\delta_{ij}+\rho_0 s^2(v_{i,j}+v_{j,i}), \tag{3.9}$$

$$v_{i,i} = 0, \tag{3.10}$$

and the second-order relations are

$$\begin{aligned}\rho_0 \ddot{v}'_i = {} & \rho_0 s^2 v'_{i,mm} - \frac{\partial p'}{\partial x_i} + v_{i,m}\frac{\partial p}{\partial x_m} \\ & -2\rho_0 \dot{v}_n \dot{v}_{i,n} + 4C_1 v_{m,n} v_{i,mn} \\ & +2C_2(v_{m,n}v_{m,ni}+v_{m,n}v_{n,mi} \\ & -v_{i,m}v_{m,nn}-v_{m,i}v_{m,nn}),\end{aligned} \tag{3.11}$$

$$\begin{aligned}t'_{ij} = {} & -p'\delta_{ij}+\rho_0 s^2(v'_{i,j}+v'_{j,i}) \\ & +2C_2(v_{m,n}v_{n,m}+v_{m,n}v_{m,n})\delta_{ij} \\ & +2C_1(v_{i,m}v_{j,m}+v_{i,n}v_{n,j}+v_{j,n}v_{n,i}) \\ & -2C_2 v_{m,i}v_{m,j},\end{aligned} \tag{3.12}$$

$$v'_{k,k} = -\tfrac{1}{2}v_{m,n}v_{n,m}, \tag{3.13}$$

where C_1 and C_2 are material constants and

$$\rho_0 s^2 = 2(C_1+C_2).$$

Finally, if we expand $U_i(X_j, t)$ and $u_i(x_j, t)$ in Taylor series about the points (x_i, t) and (X_i, t) respectively and use expansions (3.1), we obtain the following relations between the first- and second-order displacement functions of (X_i, t) and the corresponding displacement functions of (x_i, t):

$$V_i(\zeta_j, t) = v_i(\zeta_j, t), \tag{3.14}$$

$$V'_i(\zeta_j, t)-v'_i(\zeta_j, t) = \frac{\partial V_i(\zeta_j, t)}{\partial \zeta_m}V_m(\zeta_n, t) = \frac{\partial v_i(\zeta_j, t)}{\partial \zeta_m}v_m(\zeta_n, t), \tag{3.15}$$

where ζ_i is either X_i or x_i.

4. GENERAL SOLUTIONS

The first-order and the second-order equations of motion for a compressible material, (3.4) and (3.6), are of the form

$$\frac{\partial^2 Q_i}{\partial t^2}-(c^2-s^2)\frac{\partial^2 Q_m}{\partial x_m \partial x_i}-s^2\frac{\partial^2 Q_i}{\partial x_m \partial x_m} = K_i, \tag{4.1}$$

where the functions $Q_i(x_j, t)$ are to be determined and $K_i(x_j, t)$ are known. Equation (4.1) also has the vector representation

$$\frac{\partial^2 \mathbf{Q}}{\partial t^2}-(c^2-s^2)\nabla(\nabla\cdot\mathbf{Q})-s^2\nabla^2\mathbf{Q} = \mathbf{K}, \tag{4.2}$$

where $\mathbf{Q}$ is the vector with components Q_i, $\mathbf{K}$ is the vector with components K_i, ∇ is the gradient operator and ∇^2 the Laplacian operator. In these equations, Q_i stands for either the first-order or the second-order displacements. The inhomogeneous factors K_i are zero for the first-order equations and are assumed to be known for the second-order equations, since the classical results are usually completely determined before the second-order effects are sought.

Under appropriate conditions, the Helmholtz decomposition theorem shows that there exist functions $k(x_i, t)$ and $\mathbf{K}^*(x_i, t)$ such that

$$\mathbf{K} = \nabla k + \nabla \times \mathbf{K}^*, \qquad \nabla \cdot \mathbf{K}^* = 0. \tag{4.3}$$

By a straightforward extension of the proof for the case with $\mathbf{K} = 0$ (see [9]), we arrive at the result that, under suitable regularity conditions, any solution of (4.2) can be written as

$$\mathbf{Q} = \nabla q + \nabla \times \mathbf{Q}^*, \tag{4.4}$$

where q and $\mathbf{Q}^*$ satisfy the equations

$$\frac{\partial^2 q}{\partial t^2} - c^2\nabla^2 q = k, \qquad \frac{\partial^2 \mathbf{Q}^*}{\partial t^2} - s^2\nabla^2 \mathbf{Q}^* = \mathbf{K}^*, \qquad \nabla \cdot \mathbf{Q}^* = 0. \tag{4.5}$$

For the incompressible medium, we note that if we define a modified second-order displacement w_i by

$$w_i = v_i' + \tfrac{1}{2} v_n v_{i,n}, \tag{4.6}$$

the incompressibility condition (3.13) on the second-order displacements becomes

$$w_{i,i} = 0. \tag{4.7}$$

Thus, both the first-order and the second-order field equations for an incompressible material can be written in the form

$$\frac{\partial^2 \mathbf{Q}}{\partial t^2} - s^2\nabla^2 \mathbf{Q} + \nabla M = \mathbf{K}, \tag{4.8}$$

$$\nabla \cdot \mathbf{Q} = 0. \tag{4.9}$$

As before $\mathbf{K}$ is known, and $\mathbf{Q}$ and M are to be determined.

The following result can then be shown to hold (see [10]†): Under suitable regularity conditions, any solution $\mathbf{Q}$, M of equations (4.8) and (4.9) can be written in the form

$$\mathbf{Q} = \nabla q + \nabla \times \mathbf{Q}^*, \qquad M = k - \frac{\partial^2 q}{\partial t^2}, \tag{4.10}$$

where q and $\mathbf{Q}^*$ obey the relations

$$\nabla^2 q = 0, \qquad \nabla \cdot \mathbf{Q}^* = 0,$$

$$\frac{\partial^2 \mathbf{Q}^*}{\partial t^2} - s^2\nabla^2 \mathbf{Q}^* = \mathbf{K}^*,$$

with k and $\mathbf{K}^*$ as in (4.3).

† A subsequent derivation may be found in [11] for the case $\mathbf{K} = 0$.

5. ONE-DIMENSIONAL EXAMPLES

As we have seen in the two preceding sections, the second-order displacements satisfy equations which are identical in form with those for the infinitesimal motion of an elastic medium under the action of (known) body forces varying in space and time. In our case, however, the body forces or inhomogeneous elements are themselves derived from solutions to these same equations with zero body force. In certain instances, as discussed previously by several authors (see [12–15]), the body force terms may be such that resonance occurs in the second-order effects, and the amplitude of the second-order disturbance can increase indefinitely with time. In this section, we discuss results for some simple one-dimensional problems in order to illustrate the nature of second-order solutions.

For one-dimensional displacement fields, we assume that the first- and second-order quantities defined in Section 3 are functions of X_1 and t only, or of x_1 and t only, depending upon whether particle or spatial coordinates are used. For the remainder of this section, we shall use (X_1, t) as independent variables, but the analysis would proceed similarly with (x_1, t) as independent variables. The field equations for $V_i(X_j, t)$ and $V_i'(X_j, t)$ can be obtained from those for $v_i(x_j, t)$ and $v_i'(x_j, t)$ in Section 3 by using the relations (3.14), (3.15).

Our first example is concerned with a semi-infinite medium of compressible material occupying the region $X_1 \geq 0$. The body is at rest and is undeformed at time $t = 0$ and subsequently a normal pressure of known amount $\varepsilon P(t)$ is applied to the surface $X_1 = 0$, the body being undeformed at points at infinity in the positive X_1-direction. We assume that the ensuing motion is one-dimensional with no displacement perpendicular to the X_1-direction so that

$$V_2 = V_3 = V_2' = V_3' = 0. \tag{5.1}$$

The infinitesimal theory has for this problem the solution

$$\rho_0 c V_1 = \int_0^\delta P(t')\,dt', \tag{5.2}$$

in which $\delta = t - X_1/c$ and in which $P(t)$ is assumed to be zero for negative values of its argument. The displacement V_1 is then zero for $X_1 \geq ct$, as would be expected.

The solution for the second-order displacement is given by

$$\begin{aligned} \rho_0^3 c^5 V_1' &= 6\beta F(\delta) - \frac{6\beta X_1}{c} P^2(\delta) \quad \text{for} \quad 0 \leq X_1 < ct, \\ V_1' &= 0 \quad \text{for} \quad X_1 > ct, \end{aligned} \tag{5.3}$$

where $F(\delta)$ is defined by

$$F(\delta) = \int_0^\delta P^2(t')\,dt', \tag{5.4}$$

and

$$\beta = (a_2 + 2a_4). \tag{5.5}$$

If $P(t)$ is continuous for all times, the first-order displacement V_1 and its first derivatives are continuous everywhere, including the front $X_1 = ct$ of the disturbance. The second-order displacement V_1' is also continuous. Perhaps the most striking feature of the solution is the presence of the second term on the right-hand side of equation (5.3),

$$-\frac{6\beta X_1}{c}P^2(\delta). \tag{5.6}$$

This term represents a wave moving with speed c and we see that the amplitude of this wave increases linearly with distance traveled and therefore linearly with time.

The solution for the case where the applied pressure at the surface $X_1 = 0$ is a discontinuous function of time can be obtained by taking the limit of the solutions for a sequence of appropriate continuous pressure distributions. If $P(t)$ has finite jumps at $t = t_0, t_1, \ldots$, then the continuous first-order displacement has discontinuities in its first derivatives at the locations $X_1 = c(t-t_0)$, $c(t-t_1), \ldots$. The second-order displacement V'_1, however, is itself discontinuous at these locations. In practice it is unlikely that discontinuous pressures can be applied to the surface of a body, but the theoretical solution for discontinuous $P(t)$ is nevertheless of interest since it does indicate the kind of results to expect when a continuous pressure varies rapidly. A more detailed discussion of the case when the first-order displacements have sharp wave fronts is given in the next section.

We may similarly treat the second-order effects associated with a classical shear wave having continuous displacement components [10], zero initial conditions and zero boundary conditions being applied to the second-order quantities. For a compressible material, we find that a second-order dilatational wave is produced but that the second-order displacement remains continuous, even if the classical displacement has discontinuous first derivatives. In an incompressible medium, the inhomogeneous terms of the second-order field equations are zero for this case so that all second-order quantities are found to be identically zero.

6. JUMP CONDITIONS AT A SURFACE OF DISCONTINUITY

In Section 3, the equations of motion for first- and second-order displacements were obtained under the implicit assumption that their second derivatives with respect to space and time existed. According to the classical theory, however, (see [16], for example) there can be, in the material, moving surfaces across which certain first derivatives of the displacements v_i are discontinuous and where, therefore, second derivatives do not exist. Specifically, at a surface moving with the dilatational wave speed c, the normal component of the displacement vector may have discontinuities in its normal derivative and in its time derivative, while for a surface moving with the shear wave speed s, the normal and time derivatives of a tangential component can be discontinuous. Although waves with such sharp fronts are unlikely to occur in a real material, the theory is still of value because it does give a good approximation of the actual situation where the displacements vary rapidly across a narrow moving zone. A solution with discontinuous derivatives may thus be considered as the limiting form of a sequence of solutions with continuous derivatives.

When the first-order displacements have continuous first and second derivatives, the equations for the second-order displacements are identical in form with those of the classical theory involving finite body force terms. If we now consider a sequence of first-order displacements which tends to a limiting displacement field with discontinuous first derivatives, the second-order displacements are, in the limit, similar in form to those for the classical elastic body under the action of concentrated forces spread over a moving surface. Now, if concentrated forces act over a surface moving with a velocity which is *not* c or s, the infinitesimal theory of elasticity yields displacements which are continuous across the surface, but which may have discontinuous derivatives in the normal direction as well as discontinuous time derivatives at the surface. If, however, the concentrated surface forces are moving with one of the wave speeds, it is no longer possible, in general, to find a solution in which the displacement is continuous across the moving surface where the force is applied, because such a surface, in four-dimensional x_i-t space, then coincides with a characteristic surface for the equations.

Thus when the first-order solution involves a sharp dilatational wave front moving with velocity c, the concentrated force is applied at a moving surface on which we cannot specify in advance the normal derivative of the normal component of displacement. In the solution for the second-order displacements, therefore, the normal component is in general discontinuous at the moving surface and the tangential components are continuous, but with discontinuous normal and time derivatives there. On the other hand, when the the first-order solution has discontinuous derivatives associated with the propagation of a shear wave (speed s), the tangential components of the second-order displacements are generally discontinuous across the moving surface, while the normal displacement is continuous with discontinuities in its normal and time derivatives.

A similar situation is that of the vibrating string acted upon by a moving concentrated transverse load. When the load has a speed equal to the wave speed, we cannot obtain, within the framework of the linearized theory, a solution for which the displacement of the string remains continuous at the point of application of the load. If a solution is found for a load speed different from the wave speed, the limiting form of the solution, as the speed of the load approaches the wave speed, is found to involve a discontinuous deflection of the string.

In this section, we use a limiting process to derive conditions on the discontinuities in the second-order displacements and their derivatives across surfaces on which the first-order displacements are continuous, but have discontinuous derivatives. Such a surface at which the first-order displacements have discontinuities in their derivatives is denoted by S, Fig. 1(a), and we suppose that it is moving with speed κ into region 2 from region 1, the constant κ being c or s for a compressible material and s for an incompressible material. We designate by P_o a typical point on S at time t_o with coordinates x_i^o. The unit normal to the surface S at P_o is taken to be directed into region 2, that is in the direction of propagation of the surface, and is denoted by n, with components n_i, while two unit tangents there are denoted by $\underline{\mu}$ and $\underline{\nu}$, with components μ_i and ν_i, where $\underline{n}$, $\underline{\mu}$, and $\underline{\nu}$ form a right-handed orthogonal triad.

Jumps in value of quantities at the point P_o on S will be indicated by the notation

$$[Q] = Q^2 - Q^1,$$

where a superscript 1 or 2 means here the limiting value of a quantity as the surface is approached from region 1 or 2 respectively.

The first-order or classical displacements v_i are continuous across S, so that $[v_i] = 0$ there. It then follows [16] that

$$[\dot{v}_i n_j + \kappa v_{i,j}] = 0. \tag{6.1}$$

The kinematical conditions (6.1) express the continuity of the spatial derivatives of v_i in directions tangential to the surface S, as well as the relations between the normal derivative of the displacement and its time derivative.

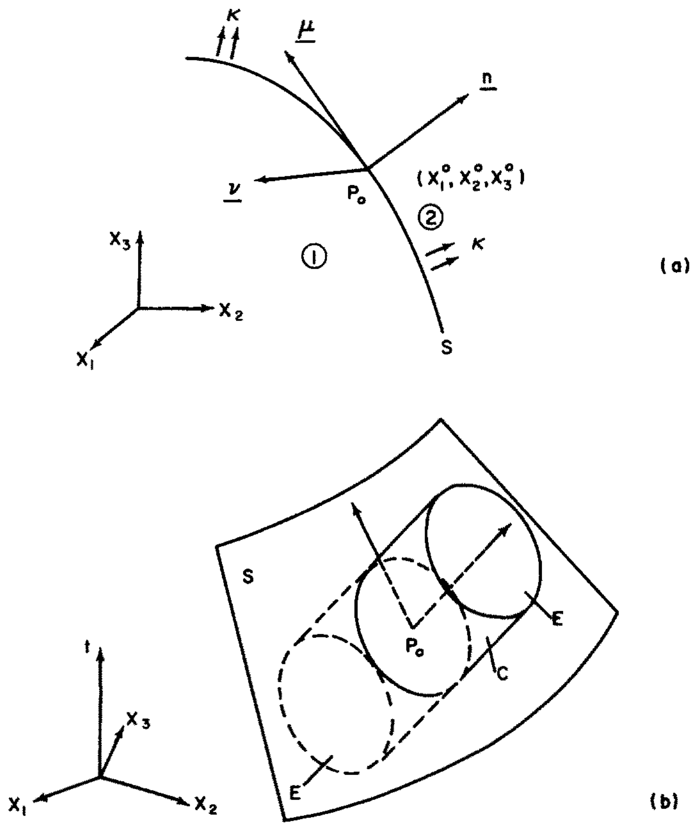

FIG. 1. Surface of discontinuity S in (a) three-dimensional (x_i) space and (b) four-dimensional (x_i, t) space.

If $\kappa = c$, which is possible only in a compressible material, the tangential components of the first-order displacement have continuous space and time derivatives, but the same does not necessarily apply for the normal component so that we have

$$[\dot{v}_i] = [\dot{v}_j n_j] n_i = j_D n_i, \tag{6.2}$$

where $j_D = [\dot{v}_i n_i]$ is a measure of the strength of the dilational discontinuity. The tangential components of the second-order displacement also are continuous, so that we have

$$\begin{aligned} [\dot{v}'_i \mu_i + c v'_{i,j} \mu_i n_j] &= 0, \\ [\dot{v}'_i \nu_i + c v'_{i,j} \nu_i n_j] &= 0. \end{aligned} \tag{6.3}$$

In addition, the continuity of the tangential components implies continuity of their derivatives in the tangential directions and we find

$$\begin{aligned} [v'_{i,j} \mu_i \mu_j] &= [v'_{i,j} \mu_i \nu_j] = 0, \\ [v'_{i,j} \nu_i \mu_j] &= [v'_{i,j} \nu_i \nu_j] = 0. \end{aligned} \tag{6.4}$$

For a shearing discontinuity, $\kappa = s$, the only discontinuities in derivatives occur in the normal and time derivatives of a tangential component of the first-order displacement.

Thus in this case $[\dot{v}_i n_i] = 0$ and we write

$$[\dot{v}_i] = [\dot{v}_j \mu_j]\mu_i = j_S \mu_i, \tag{6.5}$$

where $j_S = [\dot{v}_i \mu_i]$ measures the strength of the jump and where μ lies in the direction of the tangential component with the discontinuous derivatives. The second-order displacement has a continuous normal component and this gives

$$\begin{aligned} [\dot{v}_i' n_i + s v_{i,j}' n_i n_j] &= 0, \\ [v_{i,j}' n_i \mu_j] = [v_{i,j}' n_i \nu_j] &= 0. \end{aligned} \tag{6.6}$$

For an incompressible material, only shear waves are possible, and, in addition to (6.5), we have

$$[p] = 0. \tag{6.7}$$

The second-order equations of motion (3.6) for a compressible material can be written in the following form:

$$\begin{aligned} \rho_0 \ddot{v}_i' = \rho_0(c^2 - s^2)v_{j,ji}' + \rho_0 s^2 v_{i,jj}' - \rho_0 \frac{\partial}{\partial x_r}(\dot{v}_i \dot{v}_r) \\ + \frac{\partial s_{ij}'}{\partial x_j} - \rho_0 \frac{\partial}{\partial t}(\dot{v}_r v_{i,r} - \dot{v}_i v_{r,r}), \end{aligned} \tag{6.8}$$

where the quantities s_{ij}' stand for the terms quadratic in v_i on the right-hand side of equations (3.7). We now integrate equations (6.8) throughout a cylinder in four-dimensional (x_i, t) space, Fig. 1(b). By repeated use of the divergence theorem, each of the volume integrals arising from the various terms in (6.8) is transformed into a surface integral over the ends E or over the curved surface C of the cylinder. In this process, care is taken to transform all of the volume integrals with integrands involving derivatives normal to the surface S (in x_i-t space) into integrals over the ends E. We then consider a sequence of smooth first-order displacements which approaches a displacement field v_i with possibly discontinuous, but finite, first derivatives at the surface S. The sequence of second-order displacements then approaches a field v_i' which is finite but possibly discontinuous across S.

Finally, we divide the resulting equation by the area of the ends E and take the limit as the cylinder is shrunk down to the point P_o in such a way that the ratio of the area of the curved surface C to the area of the ends E goes to zero as the volume of the cylinder goes to zero. As a result of this analysis we obtain the following second-order jump conditions (momentum balance):

$$\begin{aligned} &\rho_0(\kappa^2 + s^2)[\dot{v}_i' + \kappa v_{i,j}' n_j] \\ &+ \rho_0(c^2 - s^2)[\dot{v}_j' n_j - \kappa v_{m,j}' n_m n_j] n_i \\ &+ 2\rho_0 \kappa[\kappa \dot{v}_i' + s^2 v_{i,j}' n_j] + 2\kappa[s_{ij}' n_j] \\ &+ 2\rho_0 \kappa(c^2 - s^2)[v_{j,j}' n_i + v_{j,i}' n_j] \\ &+ 2\rho_0 \kappa[\kappa \dot{v}_j v_{i,j} - \dot{v}_i \dot{v}_j n_j - \kappa v_{j,j} \dot{v}_i] = 0. \end{aligned} \tag{6.9}$$

These equations, together with the appropriate continuity conditions, (6.3) and (6.4) for $\kappa = c$ and (6.6) for $\kappa = s$, provide the information needed across a characteristic surface for a properly set problem. We see from (6.9) that the second-order displacement v_i' is in general discontinuous across S, as expected, because the quantity

$$\frac{\mathrm{d}}{\mathrm{d}t}[v_i'] = [\dot{v}_i'+\kappa v_{i,j}'n_j]$$

represents the rate of change of the jump in v_i' as seen by an observer at P_o moving in the normal direction with the surface S.

The implications of these results may be seen more clearly if we take a special coordinate system in x_i-space for which at time t_o

$$\underline{n} = (1,0,0), \quad \underline{\mu} = (0,1,0), \quad \underline{v} = (0,0,1). \tag{6.10}$$

Thus, at P_o, the wave front is traveling in the x_1-direction. If $\kappa = c$, $\dot{v}_1$ and $v_{1,1}$ are discontinuous. The jump conditions on v_i' may then be written

$$\begin{aligned} [\dot{v}_2'+cv_{2,1}'] &= [\dot{v}_3'+cv_{3,1}'] = 0, \\ [v_{2,2}'] = [v_{2,3}'] &= [v_{3,2}'] = [v_{3,3}'] = 0, \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} 2\rho_0c^2\frac{\mathrm{d}}{\mathrm{d}t}[v_1'] = {} & 2\rho_0cj_D\tilde{\dot{v}}_1 \\ & +4(a_1+2a_2+4a_3+12a_4)j_D(v_{2,2}+v_{3,3}) \\ & +8(5a_2+6a_4)j_D\tilde{v}_{1,1}, \end{aligned} \tag{6.12}$$

$$\begin{aligned} \rho_0c(c^2-s^2)[v_{1,2}'-v_{2,1}'] &= 4(2a_2-a_3)j_Dv_{1,2}-2(a_1+2a_3)j_Dv_{2,1}, \\ \rho_0c(c^2-s^2)[v_{1,3}'-v_{3,1}'] &= 4(2a_2-a_3)j_Dv_{1,3}-2(a_1+2a_3)j_Dv_{3,1}, \end{aligned} \tag{6.13}$$

where $\tilde{Q} = \frac{1}{2}(Q^1+Q^2)$.

Now v_1' is continuous in the undeformed state (at $t = 0$) so that $[v_1']$ is initially zero. The value of $[v_1']$ at later times can therefore be determined from (6.12), and this value can then be substituted into equations (6.13) to give $[v_{2,1}']$ and $[v_{3,1}']$. From (6.11), we can solve for $[\dot{v}_2']$ and $[\dot{v}_3']$. Thus, the jump conditions serve to determine the jump in the normal component, v_1' in this case, and the jump in the first derivatives of the continuous tangential components, v_2' and v_3' here. This information is precisely that which must be provided, in a properly set problem, on a characteristic surface of the dilatational type, in order to be able to proceed with the determination of the solution v_i' on one side of the surface when the solution is known on the other side.

In the same way, when $\kappa = s$ (see [10]), the jump conditions (6.9) together with (6.6) provide the jumps in the tangential components of the displacement and the jumps in the first derivatives of the continuous normal component.

For an incompressible material ($\kappa = s$), an analogous limiting process can be done for equation (3.11) (see [10]), and if the resulting expressions are resolved in the normal and tangential directions, we obtain

$$[p'] = 0, \tag{6.14}$$

$$2\rho_0s^2\frac{\mathrm{d}}{\mathrm{d}t}[v_i'\mu_i] = 2\rho_0sj_S\dot{v}_in_i+4C_1j_Sv_{i,j}n_in_j-4C_2v_{i,j}\mu_i\mu_j, \tag{6.15}$$

$$2\rho_0s^2\frac{\mathrm{d}}{\mathrm{d}t}[v_i'v_i] = -2C_2j_S(v_{i,j}\mu_iv_j+v_{i,j}v_i\mu_j). \tag{6.16}$$

In addition, the incompressibility equation (3.13) and equations (6.1), (6.5) yield

$$s[v'_{i,i}] = j_S v_{i,j} n_i \mu_j. \tag{6.17}$$

(It may be noted that integrating (3.13) throughout the cylindrical region of Fig. 1(b) leads to the continuity of the normal displacement component, as previously assumed.) Equations (6.14)–(6.17) together with (6.6) then form the relations needed on the characteristic surface for this case.

The discontinuities in the second-order displacements at a sharp wave front are a consequence of the assumption, in the expansions (3.1), that the displacements v_i, v'_i and V_i, V'_i are independent of the parameter ε. According to the non-linear theory, a disturbance with a sharp wave front propagates with a velocity which depends upon the strength of the disturbance, in contrast to the constant wave speeds for infinitesimal theory. Thus, the location of the actual surface across which the displacements have discontinuous derivatives depends upon ε, and it is not possible to choose v_i and v'_i (or V_i and V'_i) so as to satisfy jump conditions across the actual surface because they are independent of ε. Indeed, they can obey jump relations only across a first approximation to the actual surface, a surface S moving with speed c or s. The result of this property of the second-order theory is to provide displacements which are in error by amounts of the order of ε^2 in a zone whose width is of the order of ε moving with the surface of discontinuity. Elsewhere, the error is $o(\varepsilon^2)$ as indicated in (3.1).

To illustrate our analysis of discontinuities in second-order displacements, we consider a simple one-dimensional exact solution to the equations of motion:

$$\begin{aligned} u_1 &= -\varepsilon(x_1 - vt) \quad \text{for} \quad x_1 \le vt, \\ u_1 &= 0 \qquad\qquad\quad\ \text{for} \quad x_1 \ge vt, \\ u_2 &= u_3 = 0, \end{aligned} \tag{6.18}$$

where v is the constant exact speed of the actual wave front. If we now seek the first- and second-order results corresponding to this exact solution, we substitute expansions (3.1) into (6.18); but because first- and second-order quantities must be independent of ε, the actual surface of discontinuity, $x_1 = vt$, must be approximated by the surface S, here $x_1 = ct$, moving with the classical wave speed c. The actual speed v will differ from c by a term of the order of ε and we therefore write

$$v = c + \varepsilon c' + o(\varepsilon^2).$$

The successive approximation results are then given by

$$\begin{aligned} v_1(x_1, t) &= -(x_1 - ct) \quad \text{for} \quad x_1 \le ct, \\ v_1(x_1, t) &= 0 \qquad\qquad\quad \text{for} \quad x_1 \ge ct, \end{aligned}$$

and

$$\begin{aligned} v'_1(x_1, t) &= c't \qquad\qquad \text{for} \quad x_1 < ct, \\ v'_1(x_1, t) &= 0 \qquad\qquad\ \ \text{for} \quad x_1 > ct. \end{aligned}$$

We notice that unless $c' = 0$, $v'_1(x_1, t)$ is indeed discontinuous at $x_1 = ct$ and that the amount of the discontinuity depends directly upon $(v-c)t$ and therefore upon the distance between the approximating and the actual surface of discontinuity. From the second-order jump conditions obtained in this section, c' is found to take the value $-6\beta/\rho_0 c$, and this is in agreement with the value for v obtainable from the exact momentum jump conditions (see [1], for example).

7. A SPHERICAL WAVE EXAMPLE

As a final example, we consider a compressible medium for which the deformation has complete spherical symmetry. All quantities are measured with respect to a fixed Cartesian coordinate system x, whose origin coincides with the center of symmetry; and all quantities therefore are functions only of time t and of distance from the origin. We designate the radial displacement by $U(R, t)$, with time t and R, the radial distance of a particle in the undeformed state, taken as independent variables. The appropriate equations of motions and stress–displacement gradient relations, exact through the second order in U, then may be derived either from the general theory as given in [17], or by transformation of the Cartesian forms of Section 2.

As before, we assume that the displacement U has an expansion in ε of the form

$$U(R,t) = \varepsilon V(R,t)+\varepsilon^2 V'(R,t)+o(\varepsilon^2). \tag{7.1}$$

By using this expansion in the field equations for U, we can obtain both the classical and second-order field equations.

From the analysis of the previous section, jump conditions can be deduced. In this case, the appropriate surfaces are spherical ones centered at the origin. We omit here the details of the derivation (see [10]), but it is found for a spherical surface S extending outwards with velocity c, the jump conditions on V and V' are

$$[\dot{V}+cV_R] = 0, \tag{7.2}$$

$$2\rho_0 c[\dot{V}'+cV'_R] = -12\beta[(V_R)^2]-8(a_1+2a_2+4a_3+12a_4)[VV_R/R], \tag{7.3}$$

where

$$\dot{V} = \frac{\partial V(R,t)}{\partial t}, \qquad V_R = \frac{\partial V(R,t)}{\partial R},$$

with similar definitions for $\dot{V}'$ and V'_R, and where β is given by (5.5).

We now consider an infinite body with a spherical cavity of initial radius a. The body is undeformed until the time $t = 0$, when a pressure of amount $\varepsilon P(t)$ is applied to the cavity. The function $P(t)$ is continuous for $t \geq 0$, but not necessarily equal to zero at $t = 0$; in addition, the body remains undisturbed at infinity, with no ingoing waves from infinity. We then deduce that no disturbance occurs in the region $R > a+ct$, so that V and V' are zero there. It is then a straightforward matter to determine the classical displacement V for the domain $a \leq R < a+ct$. From this solution, we find that, if $P(0)$ is non-zero, the surface $R = a+ct$ is a sharp wave front; and by integrating the jump condition (7.3) and using the initial conditions, we arrive at the result that, as the surface $R = a+ct$ is approached from inside, V' tends to the value

$$V'(a+ct-0,t) = \frac{-6\beta a^2 P^2(0)}{\rho_0^3 c^6}\left\{\frac{1}{a}-\frac{1}{a+ct}\right\}. \tag{7.4}$$

If we let time grow very large, $V'(a+ct-0, t)$ tends towards the constant value

$$-6\beta a P^2(0)/\rho_0^3 c^6.$$

This contrasts with the results obtained for the one-dimensional dilational wave (5.3) where

$$V'_1(ct-0, t) = -\left\{\frac{6\beta P^2(0)}{\rho_0^3 c^6}\right\} ct.$$

In one case, a first-order disturbance of finite extent is spreading into an infinite region and therefore decreasing in strength, while in the other case the disturbance propagates without change of form. A similar analysis may be applied to the problem where $P(t)$ has a denumerable set of ordinary discontinuities for positive values of its argument.

The complete solution to the second-order field equations can be found in a fairly straightforward manner (see [10]), but it is important to notice that condition (7.4) is precisely that needed to determine the solution for V' in the region $a \leq R \leq a+ct$. Thus, in the general second-order theory where displacements are not continuous across wave fronts, jump conditions form an integral part of the field equations since they enable us to continue the solution across the wave fronts.

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Résumé—Des effets de second ordre dans la propagation d'ondes élastiques par des milieux isotropes homogènes sont considérés. Les conditions de discontinuité dirigeant des quantités de second ordre aux ondes enveloppantes où les déplacements de premier ordre ont des dérivatifs discontinus sont dérivés, et ils est constaté qu'une théorie consistante requiert que les déplacements de second ordre soient discontinus à ces ondes enveloppantes, en général. Les résultats sont illustrés par des exemples à une dimension mettant en cause des ondes planes et sphériques.

Zusammenfassung—Wirkungen zweiter Ordnung in der Fortpflanzung von elastischen Wellen durch homogene isotrope Medien werden betrachtet. Die Sprungbedingungen welche die Quantitäten zweiter Ordnung in den Wellenfronten bestimmen, in welchen die Verschiebungen erster Ordnung, unterbrochene Ableitungen haben, werden abgeleitet und es wird festgestellt, dass eine einheitliche Theorie erfordert, dass die Verschiebungen zweiter Ordnung in solchen Wellenfronten im allgemeinen unterbrochen werden. Die Ergebnisse werden mit einigen eindimensionalen Beispielen, welche ebene und sphärische Wellen einschliessen, illustriert.

Абстракт—Рассматриваются эффекты второго порядка в распространении эластичных волн сквозь гомогенную изотропную среду. Выведены условия скачка, регулирующие величины второго порядка у волновых фронтов, где смещения первого порядка обладают разрывными производными и видно, что совместимая теория требует в общем смещений второго порядка, чтобы быть разрывной у таких волновых фронтов. Результаты поясняются несколькими опномерными примерами, включая плоские и сферические волны.